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Attainable Sets for Linear Stochastic Control Systems*

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Let $x_t^u(w)$ be the solution process of the n-dimensional stochastic differential equation $dx_t^u = [A(t)x_t^u + B(t)\,u(t)]\,dt + C(t)\,dW_t$, where $A(t)$, $B(t)$, $C(t)$ are matrix functions, W_t is a n-dimensional Brownian motion and u is an admissable control function. For fixed $\epsilon \geqslant 0$ and $1 \geqslant \delta \geqslant 0$, we say that $x \in R^n$ is (ϵ, δ) attainable if there exists an admissable control u such that $P\{x_t^u \in S_\epsilon(x)\} \geqslant \delta$, where $S_\epsilon(x)$ is the closed ϵ-ball in R^n centered at x. The set of all (ϵ, δ) attainable points is denoted by $\mathscr{A}(t)$. In this paper, we derive various properties of $\mathscr{A}(t)$ in terms of $K(t)$, the attainable set of the deterministic control system $\dot{x} = A(t)x + B(t)u$. As well a stochastic bang-bang principle is established and three examples presented.

1. INTRODUCTION

In [1] we defined the notion of a finite dimensional attainable set for a stochastic control system and proved under rather mild general conditions that these sets are compact in R^n and continuous in time with respect to the Hausdorff metric on the compact subsets of R^n. In this paper, we shall characterize the properties of attainable sets for control systems governed by linear stochastic differential equations. In particular, we shall relate the attainable sets to the attainable sets of the deterministic part of the linear stochastic differential equation. This will lead to a direct method for computing the stochastic attainable sets.

Given a probability space $(\Omega, \mathscr{F}, P)$ with respect to which W_t is an n-dimensional Brownian motion, let us consider the n-dimensional linear stochastic differential equation on $[0, T]$,

$$dx_t^u = [A(t)\,x_t^u + B(t)\,u(t)]\,dt + C(t)\,dW_t \tag{1}$$

where $A(t)$, $C(t)$ are $n \times n$ matrices, $B(t)$ is an $n \times m$ matrix, all bounded and Lebesgue measurable on $[0, T]$, and $u \in \mathscr{U}$, the class of control functions which are Lebesgue measurable mappings from $[0, T]$ into U, a compact convex

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subset of R^m. Let $x_0 = c$, a constant, and let $\Phi(t) \equiv \Phi(t, 0)$ be the fundamental matrix of the deterministic part of (1),

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t). \tag{2}$$

It can be readily shown [2, p. 133] that the solution process x_t^u of (1) is Gaussian and has the normal distribution

$$N \Big\{ \Phi(t)\, c + \Phi(t) \int_0^t \Phi(s)^{-1} B(s)\, u(s)\, ds; \\ \int_0^t \Phi(t)\, \Phi(s)^{-1} C(s)\, C'(s)\, (\Phi(s)^{-1})'\, \Phi(t)'\, ds \Big\}, \tag{3}$$

where $'$ denotes transpose.

DEFINITION 1.1. For fixed $\epsilon \geqslant 0$, $1 \geqslant \delta \geqslant 0$, we say that the point $x \in R^n$ is (ϵ, δ) attainable (or simply attainable) at time $t \in [0, T]$ by the solution process of (1) if there exists $u \in \mathscr{U}$ such that

$$P\{x_t^u \in S_\epsilon(x)\} \geqslant \delta, \tag{4}$$

where $S_\epsilon(x)$ is the closed n-dimensional euclidean ball centered at x having radius ϵ, and $x_0 = c$.

The set of all (ϵ, δ) attainable points x at time t is denoted by $\mathscr{A}(t) = \mathscr{A}_\epsilon^\delta(t)$. Let $K(t)$ denote the attainable set at time t for the deterministic system (2). Observe then that if $C(t) = 0$ and $\epsilon = 0$, $\mathscr{A}(t)$ reduces to $K(t)$; it is as a result of this reduction that $\mathscr{A}(t)$ seems to be the natural extension of $K(t)$ to a stochastic system.

In terms of the distribution function (3) of x_t^u, equation (4) can be written as follows:

$$\int_{S_\epsilon(x)} \frac{1}{(2\pi)^{n/2} (|\, G(t)|)^{1/2}} \exp \Big\{ -\frac{1}{2} (y - m_t^u)\, G^{-1}(t)\, (y - m_t^u)' \Big\}\, dy \geqslant \delta, \tag{5}$$

where

$$G(t) \equiv \int_0^t \Phi(t)\, \Phi(s)^{-1} C(s)\, C'(s)\, (\Phi(s)^{-1})'\, \Phi(t)'\, ds$$

and

$$m_t^u \equiv \Phi(t)\, c + \Phi(t) \int_0^t \Phi(s)^{-1} B(s)\, u(s)\, ds.$$

From now on, we shall assume that for each t, $A(t)$, $B(t)$ and $C(t)$ are such that $G(t)$ is positive definite.

2. Some Results

We observe that a point x in $K(t)$ is of the form

$$x = \Phi(t)\, c + \Phi(t) \int_0^t \Phi(s)^{-1} B(s)\, u(s)\, ds,$$

where $u \in \mathscr{U}$. But this is precisely the mean of the random variable $x_t{}^u$. Let us assume that ϵ and δ are such that

$$\int_{S_\epsilon(0)} \frac{1}{(2\pi)^{n/2} (| G(t)|)^{1/2}} \exp \left\{- \frac{1}{2} yG^{-1}(t)\, y'\right\} dy > \delta. \tag{6}$$

Otherwise $\mathscr{A}(t) = \varnothing$. For $\mathscr{A}(t) \neq \varnothing$ it is easy to show that $\mathscr{A}(t)$ has interior and that $K(t) \subset \mathscr{A}(t)$, since every point in $K(t)$ is the center of some normal distribution (3).

Let us now fix $t \in [0, T]$. Let $Q(t)$ denote the set of points $x \in R^n$ satisfying

$$\int_{S_\epsilon(x)} \frac{1}{(2\pi)^{n/2} (| G(t)|)^{1/2}} \exp \left\{- \frac{1}{2} yG^{-1}(t)\, y'\right\} dy \geqslant \delta. \tag{7}$$

We assume that ϵ and δ are such that $Q(t) \neq \varnothing$ and is not a singleton.

Definition 2.1. A compact convex set $H \subset R^n$ is said to be smooth if at each point on the boundary of H there exists a unique support hyperplane to H.

Lemma 2.1. *$Q(t)$ is compact, strictly convex and smooth.*

Proof. Compactness follows directly from the definition of $Q(t)$. Let us now prove that $Q(t)$ is convex. Since $G(t)$ is a positive definite matrix,

$$g(x) = \int_{S_\epsilon(x)} \frac{1}{(2\pi)^{n/2} (| G(t)|)^{1/2}} \exp \left\{- \frac{1}{2} yG^{-1}(t)\, y'\right\} dy$$

is logarithmic concave in R^n [3, Theorem 3]. Let x_1 and $x_2 \in Q(t)$, i.e., $\exists$ two controls u_1 and $u_2 \in \mathscr{U} \ni$ for $i = 1, 2$,

$$g(x_i - m_t^{u_i}) \equiv \int_{S_\epsilon(x_i)} \frac{1}{(2\pi)^{n/2} (| G(t)|)^{1/2}} \exp \left\{- \frac{1}{2} (y - m_t^{u_i})\, G^{-1}(t)\, (y - m_t^{u_i})'\right\} dy$$
$$\geqslant \delta.$$

To prove convexity, we must show that $x_\lambda = \lambda x_1 + (1 - \lambda) x_2 \in Q(t)$, i.e., that $\exists u_\lambda \in \mathscr{U} \ni g(x - m_t^{u_\lambda}) \geqslant \delta$. To this end, let us choose $u_\lambda = \lambda u_1 + (1 - \lambda) u_2$.

The convexity of U ensures that $u_\lambda \in \mathscr{U}$. Now $m_t^{u_\lambda} = \lambda m_t^{u_1} + (1 - \lambda) m_t^{u_2}$ and

$$g(x - m_t^{u_\lambda}) = g(\lambda(x_1 - m_t^{u_1}) + (1 - \lambda)(x_2 - m_t^{u_2}))$$

$$\geqslant g^\lambda(x_1 - m_t^{u_1})\, g^{(1-\lambda)}(x_2 - m_t^{u_2}) \geqslant \delta.$$

Thus $x_\lambda \in Q(t)$ and $Q(t)$ is convex. We claim that in fact $Q(t)$ is strictly convex. If it were not, then some part of its surface would contain a straight line. This line in turn would have to stem from a region on the surface of the normal distribution which has a constant gradient. But this is impossible.

The smoothness of $Q(t)$ also follows directly from the smoothness properties of the normal distribution. If $Q(t)$ were not smooth, the corners in its boundary would have to stem from points on the surface of the normal distribution which do not have unique gradients. This is clearly impossible for the normal distribution.

In general, it is very difficult to determine the equation for the boundary of $Q(t)$. Each point r on the locus of the boundary would have to satisfy the following complicated integral equation in r:

$$\int_{S_\epsilon(r)} \frac{1}{(2\pi)^{n/2}(|G(t)|)^{1/2}} \exp\{-yG^{-1}(t)\, y'\}\, dy = \delta. \tag{8}$$

In the special case that $C(t) = \sigma(t)I$, where I is the identity matrix in R^n and $\sigma(t)$ is a continuous, positive function, the coordinate functions of the normal distribution are independent and have equal variance. Thus $Q(t)$ is an n-dimensional sphere with radius $r = r(t)$ obtained from (8).

Let us now consider the normal distribution with covariance matrix $G(t)$ and mean m_t^u at time t, where $u \in \mathscr{U}$. Each point x in the set $m_t^u + Q(t)$ satisfies the equation (5). Also, each point in $K(t)$ is of the form m_t^u for some $u \in \mathscr{U}$. Thus $\bigcup\{m_t^u + Q(t) : u \in \mathscr{U}\}$ is the set of points such that an ϵ-ball centered at each of these points has probability greater than or equal to δ; i.e., $\mathscr{A}(t) = \bigcup\{m_t^u + Q(t) : u \in \mathscr{U}\}$.

Let $\mathscr{C}$ be a compact convex set in R^n with $0 \in \mathscr{C}$. Let $\mathscr{K}$ be a compact convex set in R^n. Define

$$\mathscr{K}_\mathscr{C} = \bigcup\{x + \mathscr{C} : x \in \mathscr{K}\}.$$

LEMMA 2.2. *$\mathscr{K}_\mathscr{C}$ is convex.*

Proof. Let $\tilde{x}_1$ and $\tilde{x}_2$ be any points in $\mathscr{K}_\mathscr{C}$. Then $\exists$ points $x_1, x_2 \in \mathscr{K} \ni \tilde{x}_i \in x_i + \mathscr{C}$, $i = i, 2$. We must show that $\tilde{x}_\lambda = \lambda\tilde{x}_1 + (1 - \lambda)\tilde{x}_2 \in \mathscr{K}_\mathscr{C}$, where $0 \leqslant \lambda \leqslant 1$. Now

$$\tilde{x}_\lambda \in \lambda(x_1 + \mathscr{C}) + (1 - \lambda)(x_2 + \mathscr{C})$$

i.e.,

$$\tilde{x}_\lambda \in \lambda x_1 + (1 - \lambda)x_2 + \mathscr{C}.$$

Since $\mathscr{K}$ is convex, $\lambda x_1 + (1 - \lambda x_2 \in \mathscr{K}$, implying that $\tilde{x}_\lambda \in \mathscr{K}_\mathscr{C}$. Q.E.D.

LEMMA 2.3. *Let $\mathscr{K}$ be a compact convex set in R^n and let $\mathscr{C}$ also be a compact convex set in R^n which is smooth and such that $0 \in \mathscr{C}$. Then the set $\mathscr{K}_\mathscr{C}$ is smooth.*

Proof. Observe that $\mathscr{K}_\mathscr{C}$ is compact since both $\mathscr{K}$ and $\mathscr{C}$ are compact. Let $\tilde{x} \in \partial\mathscr{K}_\mathscr{C}$, the boundary of $\mathscr{K}_\mathscr{C}$. Then, by the definition of $\mathscr{K}_\mathscr{C}$, $\exists x \in \mathscr{K} \ni \tilde{x} \in \partial(x + \mathscr{C})$. Let Π be a support hyperplane to $\mathscr{K}_\mathscr{C}$ at $\tilde{x}$. Since $\tilde{x} \in \partial(x + \mathscr{C})$, Π is also a support hyperplane to $x + \mathscr{C}$ at $\tilde{x}$. But $x + \mathscr{C}$ is smooth since $\mathscr{C}$ is. Hence each point on its boundary must have a unique support hyperplane. Thus Π is unique and $\mathscr{K}_\mathscr{C}$ is smooth. Q.E.D.

LEMMA 2.4. *Let $\mathscr{K}$ be a compact, strictly convex set in R^n, and let $\mathscr{C}$ be a compact, strictly convex set in R^n with $0 \in \mathscr{C}$. Then $\mathscr{K}_\mathscr{C}$ is strictly convex.*

Proof. Suppose $\mathscr{K}_\mathscr{C}$ is not strictly convex. Then $\exists$ points $\tilde{x}_1$ and $\tilde{x}_2$ in $\partial\mathscr{K}_\mathscr{C} \ni \tilde{x}_\lambda = \lambda\tilde{x}_1 + (1 - \lambda)\tilde{x}_2 \in \partial\mathscr{K}_\mathscr{C}\, \forall\lambda,\ 0 \leqslant \lambda \leqslant 1$. Since $\tilde{x}_i \in \partial\mathscr{K}_\mathscr{C}$, $i = 1, 2$, and $\mathscr{C}$ is strictly convex $\exists$ distinct points x_1 and $x_2 \in \partial\mathscr{K} \ni \tilde{x}_i \in \partial(x_i + \mathscr{C})$, $i = 1, 2$. Thus,

$$\tilde{x}_\lambda \in \lambda(x_1 + \mathscr{C}) + (1 - \lambda)(x_2 + \mathscr{C}),$$

i.e.,

$$\tilde{x}_\lambda \in \lambda x_1 + (1 - \lambda)\, x_2 + \mathscr{C}.$$

But $\tilde{x}_\lambda \in \partial\mathscr{K}_\mathscr{C}$. Therefore $x_\lambda = \lambda x_1 + (1 - \lambda)x_2 \in \partial\mathscr{K}$; otherwise $\tilde{x}_\lambda \notin \partial\mathscr{K}_\mathscr{C}$. But $\mathscr{K}$ is strictly convex. Hence $x_\lambda \notin \partial\mathscr{K}$, and we have our contradiction. Q.E.D.

DEFINITION 2.2. Let $\mathscr{K}$ be a compact convex set in R^n. The set $s\mathscr{K} + b$, where $s > 0$ is a scalar and b is an n-dimensional vector, is said to be similar to $\mathscr{K}$.

LEMMA 2.5. *Let $\mathscr{K}$ be a compact convex set in R^n, and let $\mathscr{C}$ be a sphere of radius r in R^n. Then $\mathscr{K}_\mathscr{C}$ is not similar to $\mathscr{K}$ unless $\mathscr{K}$ is a sphere.*

Proof. Let $\mathscr{B}$ be the unit ball in R^n. Then $\mathscr{C} = r\mathscr{B}$. It is well known that the compact convex set $\mathscr{K}$ is uniquely determined by its support functional

$$h_\mathscr{K}(x) = \sup\{x \cdot y \mid y \in \mathscr{K}\},$$

for each $x \in R^n$, where $\cdot$ is the scalar product in R^n. To have similarly between $\mathscr{K}_\mathscr{C}$ and $\mathscr{K}$, we would need

$$\mathscr{K}_\mathscr{C} \equiv \mathscr{K} + r\mathscr{B} = s\mathscr{K} + b \tag{9}$$

for some $s > 1$ and vector b. Passing to support functions on both sides of (9), we get

$$h_\mathscr{K}(x) + r \mid x \mid = sh_\mathscr{K}(x) + b \cdot x,$$

where $\mid x \mid = (x \cdot x)^{1/2}$. This yields

$$h_\mathscr{K}(x) = \frac{r \mid x \mid}{s - 1} - \frac{b \cdot x}{s - 1}.$$

This says that

$$\mathscr{K} = \frac{r\mathscr{B}}{s - 1} - \frac{b}{s - 1},$$

i.e., $\mathscr{K}$ is the sphere with radius $r/(s - 1)$ and center $-b/(s - 1)$. Q.E.D.

Let us now return to the stochastic control system (1).

THEOREM 2.1. *$\mathscr{A}(t)$ is compact, continuous in t with respect to the Hausdorff metric, convex and smooth.*

Proof. Since U is compact, $K(t)$ is compact [4, p. 69]. $Q(t)$ is also compact. From this it is easy to show that

$$\mathscr{A}(t) = \bigcup \{x + Q(t) : x \in K(t)\}$$

is also compact.

Let $K(t_n) \to K(t)$ in the Hausdorff metric. This implies that $m_{t_n}^u \to m_t^u$ uniformly in $u \in \mathscr{U}$. Now the normal distribution $N(0, G(t))$ is continuous in t in the $\mathscr{L}_1$ sense on compact subsets of R^n. This, with the compactness of $Q(t)$ implies that $Q(t_n) \to Q(t)$ in the Hausdorff metric, i.e., $Q(t_n) \to Q(t)$ uniformly. Hence

$$m_t^{u_n} + Q(t_n) \to m_t^u + Q(t)$$

in the Hausdorff metric uniformly in u. Hence

$$\mathscr{A}(t_n) = \bigcup \{m_{t_n}^u + Q(t_n) : u \in \mathscr{U}\} \to \bigcup \{m_t^u + Q(t) : u \in \mathscr{U}\} = \mathscr{A}(t)$$

in the Hausdorff metric.

Since U is compact, $K(t)$ is convex [4, p. 69]. In light of Lemmas 2.2 and 2.3, $\mathscr{A}(t)$ is convex and smooth. Q.E.D.

THEOREM 2.2. *If the deterministic system* (2) *is normal, $\mathscr{A}(t)$ is strictly convex.*

Proof. The normality of the system (2) ensures that $K(t)$ is strictly convex [4]. Lemma 2.4 completes the proof. Q.E.D.

When considering a target to be hit by a stochastic control system we think of a set of points such that an ϵ-ball around each point can be hit with at least probability δ. Just as in the deterministic theory, it is important to characterize the boundary points of $\mathscr{A}(t)$ because it is with these points that the attainable set will first make contact with the target. The boundary points of $K(t)$ are induced by the class of controls $\bar{\mathscr{U}} \subset \mathscr{U}$ which satisfy the maximal equation

$$\eta(s)\, B(s)\, u(s) = \max_{u \in U} \eta(s)\, B(s)\, u \tag{10}$$

a.e. on $[0, T]$, where $\eta(s)$ is the solution of the adjoint equation $\dot{\eta}(t) = -\eta(t)\, A(t)$.

Note that the points on the boundary of $\mathscr{A}(t)$ are points an ϵ-ball around which has probability equal to δ under a normal distribution centered on the boundary of $K(t)$. Since a point x is on the boundary of $K(t)$ iff it is induced by a control satisfying the maximal equation (9), we have:

THEOREM 2.3. *The point x is on the boundary of $\mathscr{A}(t)$ iff the control u in the normal distribution* (3) *satisfies the maximal equation* (10).

Let $U^0 = [-1, 1]^m$. Recall from the deterministic theory that $K^0(t) = K(t)$, where $K^0(t)$ is the set of points attained by solutions of (2) at time t, where $U = U^0$, i.e., the class of bang-bang controls $\mathscr{U}^0$. This is referred to as the bang-bang principle. From this, we immediately obtain the following stochastic bang-bang principle:

THEOREM 2.4. *$\mathscr{A}(t) = \mathscr{A}^0(t)$, where $\mathscr{A}^0(t)$ is the set of all (ϵ, δ) attainable points with u restricted to the class of bang–bang controls $\mathscr{U}^0$.*

Proof. Let $\mathscr{A}(t) = K(t) \cup R(t)$ and $\mathscr{A}^0(t) = K^0(t) \cup R^0(t)$. Let x be any point in $R(t)$, i.e., $\exists u \in \mathscr{U} \ni$

$$\int_{S_\epsilon(x)} \frac{1}{(2\pi)^{n/2} (|G(t)|)^{1/2}} \exp\{-(y - m_t^u)\, G^{-1}(t)\, (y - m_t^u)'\}\, dy \geqslant \delta. \tag{11}$$

By virtue of the deterministic bang–bang principle, we can find $\bar{u} \in \mathscr{U}^0 \ni m_t^{\bar{u}} = m_t^u$. Hence (11) is valid for m_t^u replaced by m_t^u and $x \in R^0(t)$. Since $R^0(t) \subset R(t)$, this establishes the equality of $R^0(t)$ and $R(t)$, and completes the proof. Q.E.D.

It is of interest to see if the shape of $\mathscr{A}(t)$ can be related to the shape of $K(t)$. Even when $Q(t)$ is a sphere, this is true only in a very special case.

THEOREM 2.5. *Let $C(t) = \sigma(t)I$, where $\sigma(t) > 0$ and I is the identity matrix in R^n. Then $\mathscr{A}(t)$ is similar to $K(t)$ only if $K(t)$ is a sphere.*

Proof. Observe that the symmetry of the normal distribution implies that $Q(t)$ is a sphere in R^n for each t. The rest follows from Lemma 2.5. Q.E.D.

3. Some Examples in R^2

We shall consider below a number of two dimensional systems (1) with $C(t) = \sigma(t)I$, $\sigma(t) > 0$ and I the identity matrix in R^2. In this case, $Q(t) = r(t)\mathscr{B}$, where $\mathscr{B}$ is the unit ball in R^2 and $r(t)$ is a number > 0. Again we fix $t > 0$ and let $r = r(t)$. Let $y = f(x)$ be the equation describing the boundary of $K(t)$, the attainable set of the 2-dimensional deterministic linear system $\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$ starting at $t = 0$. Let $(x, f(x))$ be a point on $\partial K(t)$, and let $(\tilde{x}, \tilde{y})$ denote the point distance r from $(x, f(x))$ along the normal vector to $y = f(x)$; i.e., (x, y) is on the boundary of $\mathscr{A}(t)$. With the aid of analytic geometry it is easy to show that

$$\begin{aligned}\tilde{x} &= x - \frac{rf'(x)}{(1 + (f'(x))^2)^{1/2}},\\ \tilde{y} &= f(x) + \frac{r}{(1 + (f'(t))^2)^{1/2}},\end{aligned} \tag{12}$$

for $(x, f(x))$ in the first and second quadrants. When $(x, f(x))$ is in the third or fourth quadrant, we obtain:

$$\begin{aligned}\tilde{x} &= x + \frac{rf'(x)}{(1 + (f'(x))^2)^{1/2}},\\ \tilde{y} &= f(x) - \frac{r}{(1 + (f'(x))^2)^{1/2}}.\end{aligned} \tag{13}$$

If $K(t)$ is known parametrically, say $x = g(\theta)$ and $y = h(\theta)$, then we let $x = g(\theta)$ and $f(x) = h(\theta)$. Noting that $f'(x) = h'(\theta)/g'(\theta)$, and substituting into (12) and (13), we get the following parametric equations for the boundary of the stochastic attainable set $\mathscr{A}(t)$:

$$\begin{aligned}\tilde{x} &= g(\theta) \mp \frac{rh'(\theta)}{((g'(\theta))^2 + (h'(\theta))^2)^{1/2}} \frac{g'(\theta)}{|g'(\theta)|},\\ \tilde{y} &= h(\theta) \pm \frac{rg'(\theta)}{((g'(\theta))^2 + (h'(\theta))^2)^{1/2}} \frac{g'(\theta)}{|g'(\theta)|},\end{aligned} \tag{14}$$

where the quadrants determine the signs as in (12) and (13).

Example 1 (Parabolas). Let us consider the linear stochastic control system

$$dx_t = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x_t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)\right] dt + \sigma\, dW_t, \tag{15}$$

where $\sigma > 0$, $x_0 = c$, and $U = [-1, 1]$. Let $K(t)$ denote the attainable set of

$$\dot{x}_t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x_t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t). \tag{16}$$

Direct application of the maximal equation yields the following parameterization of the $\partial K(t)$:

$$\partial K(t) = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} c \pm \begin{pmatrix} \theta^2 - 2t\theta + \frac{1}{2}t^2 \\ t - 2\theta \end{pmatrix} \,\middle|\, 0 \leqslant \theta \leqslant t \right\}. \tag{17}$$

Let

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

For the moment consider the positive sign in (17) and let $g(\theta) = z_2(t) + t - 2\theta$, $h(\theta) = z_1(t) + \theta^2 - 2t\theta + \frac{1}{2}t^2$. Then $h'(\theta) = 2(\theta - t)$ and $g'(\theta) = -2$, and using (14), we have for the third and fourth quadrants:

$$\begin{aligned} \tilde{x} &= z_2(t) + t - 2\theta + \frac{2r(\theta - 2)}{(2(\theta - t)^2 + (-2)^2)^{1/2}} \cdot \frac{-2}{|-2|}, \\ \tilde{y} &= z_1(t) + \theta^2 - 2t\theta + \frac{1}{2}t^2 - \frac{2r}{(2(\theta - t)^2 + (-2)^2)^{1/2}} \cdot \frac{-2}{|-2|}. \end{aligned} \tag{18}$$

On the other hand, taking the negative sign in (17) would yield for the first and second quadrants:

$$\begin{aligned} \tilde{x} &= z_2(t) - t + 2\theta - r\frac{(-2)(\theta - t)}{(2(\theta - t)^2 + (-2)^2)^{1/2}} \cdot \frac{2}{|2|}, \\ \tilde{y} &= z_1(t) - \theta^2 + 2t\theta - \frac{1}{2}t^2 + \frac{2r}{(2(\theta - t)^2 + (-2)^2)^{1/2}} \cdot \frac{2}{|2|}. \end{aligned} \tag{19}$$

Equations (18) and (19) can be conveniently combined as follows:

$$\begin{pmatrix} \tilde{y} \\ \tilde{x} \end{pmatrix} = z(t) \pm \begin{pmatrix} \theta^2 - 2t\theta + \dfrac{1}{2}t^2 - \dfrac{r}{((\theta - t)^2 + 1)^{1/2}} \\ t - 2\theta - \dfrac{r(\theta - t)}{((\theta - t)^2 + 1)^{1/2}} \end{pmatrix}.$$

It only remains to determine the boundary of $\mathscr{A}(t)$ corresponding to the corners of the intersecting parabolas. Putting $\theta = 0$ in $h(\theta)$ and $g(\theta)$, we obtain the following center for the circle S_1:

$$z(t) + \begin{pmatrix} \frac{1}{2}t^2 \\ t \end{pmatrix}.$$

The slope on $\partial K(t)$ is $h'(\theta)/g'(\theta) = -(\theta - t)$. Thus, at the corner for S_1, the slope of the upper parabola is $-(t - t) = 0$, and for the lower parabola it is t. Therefore S_1 and the boundary of $\mathscr{A}(t)$ corresponding to the upper parabola meet at a point which makes an angle $\pi/2$ with the center of S_1. The angle made with the lower parabola is $-\tan^{-1}(1/t)$. Thus $\tilde{S}_1$, the arc of S_1 that forms part of $\partial\mathscr{A}(t)$ is given as follows:

$$\tilde{S}_1 = \left\{ z(t) + \begin{pmatrix} \frac{1}{2}t^2 + r\sin\theta \\ t + r\cos\theta \end{pmatrix} \,\middle|\, -\tan^{-1}(1/t) \leqslant \theta \leqslant \frac{\pi}{2} \right\}.$$

Symmetry yields the arc $\tilde{S}_2$ of the circle S_2 centered at the other corner:

$$\tilde{S}_2 = \left\{ z(t) - \begin{pmatrix} \frac{1}{2}t^2 + r\sin\theta \\ t + r\cos\theta \end{pmatrix} \,\middle|\, -\tan^{-1}(1/t) \leqslant \theta \leqslant \frac{\pi}{2} \right\}.$$

Thus,

$$\mathscr{A}(t) = \left\{ z(t) \pm \begin{pmatrix} \theta^2 - 2t\theta - r((\theta - t)^2 + 1)^{-1/2} \\ t - 2\theta - r(\theta - t)\,((\theta - t)^2 + 1)^{-1/2} \end{pmatrix} \,\middle|\, 0 \leqslant \theta \leqslant t \right\}$$

$$\bigcup \left\{ z(t) \pm \begin{pmatrix} \frac{1}{2}t^2 + r\sin\theta \\ t + r\cos\theta \end{pmatrix} \,\middle|\, -\tan^{-1}(1/t) \leqslant \theta \leqslant \frac{\pi}{2} \right\}.$$

EXAMPLE 2 (Hyperbolas). Consider the system

$$dx_t = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x_t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \right] dt + \sigma\, dW_t, \tag{20}$$

where $\sigma > 0$, $x_0 = c$, and $U = [-1, 1]$. Let $z(t) = e^{At}c$, where $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then using the maximal equation, we get:

$$\partial K(t) = \left\{ z(t) \pm \begin{pmatrix} 2\sinh(t - \theta) - \sinh t \\ 2\cosh(t - \theta) - (\cosh t + 1) \end{pmatrix} \,\middle|\, 0 \leqslant \theta \leqslant t \right\}, \tag{21}$$

where $z(t) = e^{At}c$.

Let us consider the first and second quadrants, and let $h(\theta) = z_1(t) + 2$ (sinh $(t - \theta)) - \sinh t$, $g(\theta) = z_2(t) + 2\cosh(t - \theta) - (\cosh t + 1)$. Then $g'(\theta) = -2\sinh(t - \theta) \leqslant 0$ for $\theta \leqslant t$ and $h'(\theta) = -2\cosh(t - \theta)$. Using (14), we obtain:

$$\tilde{y} = h(\theta) + r\,\frac{(-2\sinh h(t - \theta))}{(4\sinh^2(t - \theta) + 4\cosh^2(t - \theta))^{1/2}}\,(-1),$$

$$\tilde{x} = g(\theta) - r\,\frac{(-2\cosh(t - \theta))}{(4\sinh^2(t - \theta) + 4\cosh^2(t - \theta))^{1/2}}\,(-1).$$

Thus, combining, we have

$$\begin{pmatrix}\tilde{y}\\ \tilde{x}\end{pmatrix} = z(t) + \begin{pmatrix} 2\sinh(t-\theta) - \sinh t + r\sinh(t-\theta)\operatorname{sech} 2(t-\theta) \\ 2\cosh(t-\theta) - (\cosh t + 1) - r\cosh(t-\theta)\operatorname{sech} 2(t-\theta)\end{pmatrix}.$$

By symmetry, we can find $\binom{\tilde{y}}{\tilde{x}}$ for the third and fourth quadrants. Let S_1 and S_2 denote the circles which are centered at the corners of the intersecting hyperbolas given by (21). The center of S_1 is $z(t) + \binom{\sinh t}{\cosh t - 1}$. The slope of $\partial K(t)$ at this corner is $h'(\theta)/g'(\theta)\,|_{\theta=0} = \coth(t-\theta)|_{\theta=0} = \coth t$. Thus, the arc of S_1 that forms part of the boundary of $\mathscr{A}(t)$ is

$$\tilde{S}_1 = z(t) + \left\{\begin{matrix}\sinh t + r\sin\theta \\ \cosh t - 1 + r\cos\theta\end{matrix}\ \middle|\ 0 \leqslant \theta \leqslant \frac{\pi}{2} + \tan^{-1}(\coth t)\right\}.$$

By symmetry, we can find $\tilde{S}_2$. Combining, we have

$$\begin{aligned}\partial\mathscr{A}(t) = {}& \left\{ z(t) \pm \begin{pmatrix}\sinh\theta(2 + r/\cosh 2\theta) - \sinh t \\ \cosh\theta(2 - r/\cosh 2\theta) - \cosh t - 1\end{pmatrix}\ \middle|\ 0 \leqslant \theta \leqslant t\right\} \\ & \bigcup \left\{ z(t) \pm \begin{pmatrix}\sinh t + r\sin\theta \\ \cosh t - 1 + r\cos\theta\end{pmatrix}\ \middle|\ 0 \leqslant \theta \leqslant \tan^{-1}(\coth t)\right\}.\end{aligned}$$

EXAMPLE 3 (Ellipse). Consider the stochastic system

$$dx_t = \left[\begin{pmatrix} 0 & a/b \\ -b/a & 0\end{pmatrix} x_t + \begin{pmatrix}0\\1\end{pmatrix} u(t)\right] dt + \sigma\, dW_t, \tag{22}$$

where $\sigma > 0$, $x_0 = c$ and $U = [-1, 1]$. Application of the maximal equation yields:

$$\partial K(t) = \{X(t, \theta, n) \mid t \leqslant \theta \leqslant \pi\} \bigcup \{X(t, \theta, n-1) \mid 0 \leqslant \theta \leqslant t'\},$$

where $n\pi < t < (n+1)\pi$, $t' = t - n\pi$ and

$$x(t, \theta, n) = z(t) \pm \begin{pmatrix}(a/b)\,(2n\cos(t-\theta) - \cos t + (-1)^n) \\ 2n\sin(t-\theta) + \sin t\end{pmatrix}, \tag{23}$$

and $z(t) = e^{At}c$,

$$A = \begin{pmatrix} 0 & a/b \\ -b/a & 0\end{pmatrix}.$$

Let

$$\begin{aligned} g(\theta) &= z_1(t) - \frac{a}{b}(\cos t - (-1)^n) + \frac{2an}{b}\cos(t-\theta), \\ h(\theta) &= z_2(t) + \sin t + 2n\sin(t-\theta). \end{aligned} \tag{24}$$

Taking the positive sign in (23) and substituting (24) into (14), we get:

$$\tilde{x} = g(\theta) + \frac{rb\cos(t-\theta)}{(a^2\sin^2(t-\theta) + b^2\cos^2(t-\theta))^{1/2}},$$

$$\tilde{y} = h(\theta) + \frac{ra\sin(t-\theta)}{(a^2\sin^2(t-\theta) + b^2\cos^2(t-\theta))^{1/2}}.$$

By symmetry, we obtain $\binom{\tilde{x}}{\tilde{y}}$ for (23) with the negative sign. Combining we have:

$$\mathscr{A}(t) = \left\{ z(t) \pm \begin{pmatrix} \dfrac{a}{b}\,[2n\cos(t-\theta) - \cos t + (-1)^n] \\ \qquad + rb\cos(t-\theta)\,[a^2\sin^2(t-\theta) + b^2\cos^2(t-\theta)]^{-1/2} \\ 2n\sin(t-\theta) + \sin t \\ \qquad + ra\sin(t-\theta)\,[a^2\sin^2(t-\theta) + b^2\cos^2(t-\theta)]^{-1/2} \end{pmatrix} \right.$$

$$\left. \begin{array}{|l} t' = t - m\pi,\ m\pi < t < (m+1)\,\pi, \\ n \times \theta \in m \times (t', \pi) \cup (m+1) \times (0, t') \end{array} \right\}.$$

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